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A Short Proof of Martindale's Theorem on GPIs

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We give a short proof of the following theorem of Martindale: If a prime ring R satisfies a nontrivial generalized polynomial identity, then the ring RC, where C is the extended centroid of R, contains a minimal idempotent e such that $eRCe$ is finite dimensional over C.

Throughout the sequel, R will denote a prime ring and C its extended centroid (see [1] for the definition). Let $X = \{x_1, x_2, ...\}$ be an infinite set of *noncommuting* indeterminates $x_1, x_2, ...$. Let $C\{X\}$ be the free C-algebra in the indeterminates of X. Let $U = RC$. Consider $U\{X\} = U *_C C\{X\}$, the free product of U and $C\{X\}$ over C. Elements of $U\{X\}$ are called *generalized polynomials* (nonzero elements of $U\{X\}$ being called *nontrivial* generalized polynomials). For a generalized polynomial f, the expression $f = 0$ is said to be a *generalized polynomial identity* (abbreviated as GPI) of R if f vanishes identically under all evaluations of its indeterminates in R. The following useful theorem was proved in [1]:

THEOREM. *Assume that R is a prime ring with the extended centroid C. If R satisfies a nontrivial GPI, then RC contains a minimal idempotent e such that $eRCe$ is finite dimensional over C.*

Our aim here is to give a short and simple proof of this Theorem. The advantage of generalized polynomials is that the evaluation of a generalized poloynomial in some of its indeterminates still results in a generalized polynomial. But in doing the evaluation of a given generalized polynomial in some of its indeterminates, one should be careful about whether the generalized polynomial thus obtained is trivial or not. For this purpose, we state a commonly used criterion for testing nontriviality of generalized

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polynomials. It is implicitly used in Martindale's proof and will be exploited in our proof here as well.

Assume that R is a prime ring with the extended centroid C. Let f be a generalized polynomial with its coefficients in RC. Choose arbitrarily a C-independent subset $\mathscr{B}$ of RC such that $\mathscr{B}$ linearly spans all the coefficients of f over C. By a *$\mathscr{B}$-monomial*, we mean a generalized polynomial of the form $b_1x_1b_2x_2\cdots b_nx_nb_{n+1}$, where $b_1, ..., b_n, b_{n+1}\in\mathscr{B}$ and where $x_1, ..., x_n\in X$. Express each coefficient of f as a C-linear combination of elements in $\mathscr{B}$ and expand the resulting expression by using the distributive law. The given generalized polynomial can then be written as a C-linear combination of $\mathscr{B}$-monomials and the generalized polynomial f is nontrivial if and only if the coefficients of $\mathscr{B}$-monomials in this expression are not all vanishing. (Equivalently, this says that the expression of a generalized polynomial as a C-linear combination of $\mathscr{B}$-monomials is unique.)

A linear generalized polynomial $l(x)$ in the indeterminate x assumes the form $l(x)=\sum_{i=1}^n a_ixb_i$. By our criterion for testing nontriviality, the linear generalized polynomial $l(x)=\sum_{i=1}^n a_ixb_i$ is nontrivial if and only if $\sum_{i=1}^n a_i\otimes_C b_i\neq 0$. Equivalently, a linear generalized polynomial $l(x)$, in the indeterminate x, is nontrivial if and only if it can be written in the form $\sum_{i=1}^n a_ixb_i$, where a_i $(i=1, ..., n)$ are C-independent and where b_i $(i=1, ..., n)$ are *not* all vanishing. As in Martindale's proof, we shall have use for the following lemma ([1, Theorem 2] or [2, Lemma 1.3.3]) as our induction basis:

LEMMA. *Assume that R is a prime ring with the extended centroid C. Let $l(x)$ be a nontrivial linear generalized polynomial with the coefficients in RC and in the indeterminate x only. If the set $\{l(r): r\in R\}$ or, equivalently, the C-subspace $\{l(r): r\in RC\}$ spanned by the set $\{l(r): r\in R\}$, is finite C-dimensional, then RC contains a minimal idempotent e such that $eRCe$ is finite C-dimensional.*

Proof of the Theorem. By linearization, we may assume that R satisfies a nontrivial multilinear GPI $f(x_1, ..., x_n)=0$ in the n indeterminates $x_1, ..., x_n$. We proceed by induction on the degree n of f. If $n=1$, we are done by the Lemma above (actually, any linear generalized polynomial identity of R must be trivial by [2, Lemma 1.3.1]). So we may assume $n>1$. Choose a (finite) basis $\mathscr{B}$ for the C-subspace spanned by the coefficients of f and express f as a C-linear combination of $\mathscr{B}$-monomials. For brevity, we denote the ordered $(n-1)$-tuples $(b_1, ..., b_{n-1})$, where $b_1, ..., b_{n-1}\in\mathscr{B}$, by the vector notation $\mathbf{b}=(b_1, ..., b_{n-1})$, and let

$$\mathscr{B}^{n-1}=\times^{n-1}\mathscr{B}=\overbrace{\mathscr{B}\times\cdots\times\mathscr{B}}^{n-1\text{ times}}$$

be the set consisting of all such ordered $(n-1)$-tuples $\mathbf{b}$. In the expression of f as a C-linear combination of $\mathscr{B}$-monomials, we collect those $\mathscr{B}$-monomials starting with $x_1, ..., x_n$ occurring in this order according to their first $n-1$ coefficients $b_1, ..., b_{n-1}$. The generalized polynomial f can then be written as

$$f(x_1, ..., x_n) = \sum_{\substack{\mathbf{b} \in \mathscr{B}^{n-1} \\ \mathbf{b} = (b_1, ..., b_{n-1})}} b_1 x_1 \cdots b_{n-1} x_{n-1} l_{\mathbf{b}}(x_n) + g(x_1, ..., x_n),$$

where $l_{\mathbf{b}}(x_n)$ is a linear generalized polynomial in the indeterminate x_n only and where $g(x_1, ..., x_n)$ is the sum of those $\mathscr{B}$-monomials with the indeterminates $x_1, ..., x_n$ occurring in an order distinct from the order $x_1, ..., x_n$. Since f is nontrivial, by reindexing $x_1, ..., x_n$ if necessary, we may assume that $l_{\mathbf{b}'}(x_n)$ is nontrivial for some $\mathbf{b}' = (b'_1, ..., b'_{n-1}) \in \mathscr{B}^{n-1}$. Fix such $\mathbf{b}' = (b'_1, ..., b'_{n-1}) \in \mathscr{B}^{n-1}$. If the set $\{l_{\mathbf{b}'}(r) : r \in R\}$ is a subset of the C-linear span of $\mathscr{B}$, then the set $\{l_{\mathbf{b}'}(r) : r \in R\}$ is finite C-dimensional and we are done by the Lemma above. So we may assume that there exists $r \in R$ such that $l_{\mathbf{b}'}(r)$ is *not* in the C-linear span of $\mathscr{B}$. Fix such $r \in R$ and expand $\mathscr{B} \cup \{l_{\mathbf{b}'}(r)\}$ into a maximal independent subset $\mathscr{B}'$ of the coefficients of $f(x_1, ..., x_{n-1}, r)$ and $f(x_1, ..., x_n)$. Express $f(x_1, ..., x_{n-1}, r)$ as a C-linear combination of $\mathscr{B}'$-monomials.

We *claim* that in this expression of $f(x_1, ..., x_{n-1}, r)$ as the C-linear combination of $\mathscr{B}'$-monomials, the coefficient of the $\mathscr{B}'$-monomial

$$\mu_0 = b'_1 x_1 \cdots b'_{n-1} x_{n-1} l_{\mathbf{b}'}(r)$$

is 1. First, for any $\mathbf{b} = (b_1, ..., b_{n-1}) \in \mathscr{B}^{n-1}$, $b_1 x_1 \cdots b_{n-1} x_{n-1} l_{\mathbf{b}}(r)$ is a C-linear combination of the $\mathscr{B}'$-monomials of the form $b_1 x_1 \cdots b_{n-1} x_{n-1} b'$ for some $b' \in \mathscr{B}'$. But if $\mathbf{b} \neq \mathbf{b}'$, that is, if $b_i \neq b'_i$ for some $i = 1, ..., n-1$, then for any $b' \in \mathscr{B}'$, the $\mathscr{B}'$-monomial $b_1 x_1 \cdots b_{n-1} x_{n-1} b'$ is distinct from the $\mathscr{B}'$-monomial μ_0 and hence the expansion of $b_1 x_1 \cdots b_{n-1} x_{n-1} l_{\mathbf{b}}(r)$ as the C-linear combination of $\mathscr{B}'$-monomials *cannot* contribute to give the $\mathscr{B}'$-monomial μ_0.

Second, in $g(x_1, ..., x_n)$, only the $\mathscr{B}$-monomials of the form

$$v(x_1, ..., x_n) = a_1 x_1 \cdots a_{i-1} x_{i-1} a' x_n a'' x_i a_i \cdots x_{n-1} a_{n-1},$$

where $a_1, ..., a_{i-1}, a', a'', a_i, ..., a_{n-1} \in \mathscr{B}$, can contribute to give terms with $x_1, ..., x_{n-1}$ occurring in this order after the substitution of x_n by r. But after the substitution of x_n by r in such a $\mathscr{B}$-monomial $v(x_1, ..., x_n)$,

$$v(x_1, ..., x_{n-1}, r) = a_1 x_1 \cdots a_{i-1} x_{i-1} a' r a'' x_i a_i \cdots x_{n-1} a_{n-1}$$

is obviously a C-linear combination of the $\mathscr{B}'$-monomials of the form

$$\mu = a_1 x_1 \cdots a_{i-1} x_{i-1} b' x_i a_i \cdots x_{n-1} a_{n-1},$$

where $b' \in \mathscr{B}'$. The last coefficients a_{n-1} of each such $\mathscr{B}'$-monomial μ displayed above belong to $\mathscr{B}$ and cannot be equal to $l_{\mathbf{b}'}(r)$, which is assumed to be independent of $\mathscr{B}$. Hence the expansion of $g(x_1, ..., x_{n-1}, r)$ as a linear combination of $\mathscr{B}'$-monomials also *cannot* contribute to give the $\mathscr{B}'$-monomial μ_0. Thus in the above displayed expression of $f(x_1, ..., x_n)$, only the term

$$b'_1 x_1 \cdots b'_{n-1} x_{n-1} l_{\mathbf{b}'}(x_n)$$

can contribute to give the $\mathscr{B}'$-monomial μ_0 after the substitution of x_n by r. Consequently the coefficient of the $\mathscr{B}'$-monomial μ_0 in the expansion of $f(x_1, ..., x_{n-1}, r)$ is 1, as claimed.

It follows that the generalized polynomial $f(x_1, ..., x_{n-1}, r)$ is nontrivial and obviously gives rise to a nontrivial GPI of R. Since $f(x_1, ..., x_{n-1}, r)$ is of degree $n-1$, our proof by induction is complete.

References

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